PRESSURE DISTRIBUTION ON A WEDGE ACCELERATED IMPULSIVELY AT A SUPERSONIC MACH NUMBER

DECEMBER 28, 1962 **DOUGLAS REPORT SM-42649**

MISSILE & SPACE SYSTEMS DIVISION
DOUGLAS AIRCRAFT COMPANY, INC.
SANTA MONICA CALIFORNIA

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ABSTRACT

The nonsteady flow equations associated with an impulsively accelerated supersonic wedge were transformed into a time-independent system. This system was then linearized with respect to the velocity change. It was shown that for certain regions of the flow field a velocity potential could be defined, and a formal series solution for the potential in these regions was found. It was also shown that the pressure perturbation satisfies Laplace's equation, in a suitably transformed space, for both rotational and irrotational regions. Finally, for the special case of a small wedge angle, the coefficients in the velocity potential series expansion were determined and a closed-form expression for the pressure field was obtained.

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NOMENCLATURE

Symbol	
a	Acoustic speed
C_p	Specific heat at constant pressure
n	Direction normal to a streamline
p	Pressure
r	$\sqrt{\eta_1^2 + \eta_2^2}$
t	Time
$U_1\ U_2$	Velocity components in X_1, X_2 plane
V	Initial wedge velocity
ΔV	Increment in wedge velocity
ν_i	Dimensionless perturbation velocity
W_i	Reference Mach number
X_1, X_2	Physical space coordinates
α	Wedge angle
β	Shock angle relative to wedge velocity
γ	Ratio of specific heats
η_i	$\xi_i - W_i$

NOMENCLATURE

Symbol

θ	$\tan^{-1} \dfrac{\eta_2}{\eta_1}$
λ	Dimensionless perturbation pressure
ξ_1	$\dfrac{x_1}{a_1 t}$
ρ	Density
σ	Dimensionless perturbation density
ϕ	Velocity potential

1. INTRODUCTION

Current design philosophy in missiles and space craft has generated an interest in the effects of high longitudinal accelerations. The effects of lateral accelerations have been studied rather extensively (Refs. 1, 2, 3), particularly within the framework of linearized theory. However, little has been done to determine longitudinal effects.

Dimensional considerations and order of magnitude studies indicate (Ref. 1) that small values of the parameter ql/V^2, where q is acceleration, l is characteristic length, and V is velocity, should correspond to small acceleration effects. Since boost accelerations of the order of several hundred g are contemplated for existing designs, and since these high accelerations will prevail through the low supersonic end of the speed range, a quantitative measure of the aerodynamic effects is desirable.

This study attempts to provide insight into the flow fields associated with accelerating bodies by considering the case of a two-dimensional wedge which undergoes an instantaneous but small velocity change from one supersonic speed to another.

Following Spitzer (Ref. 4), the nonsteady flow equations for this problem are transformed into a time-independent system. A linearization of the resulting equations is effected by ignoring terms of second and higher orders in the velocity change, ΔV, considering the flow field as a perturbation from the steady attached shock wedge-flow case. The resulting linear system is then examined from the standpoint of possible solution, including the derivation of a potential equation for the case of irrotational flow. It is shown that for the general case the pressure is a solution to Laplace's equation in a suitably transformed space, and solutions for the special case of a small wedge angle are obtained.

2. WAVE GEOMETRY

Consider a wedge (Fig. 1) with initial velocity V which is increased instantaneously at $t=0$ to $V+\Delta V$, with

$$\frac{\Delta V}{V} \ll 1$$

We choose a coordinate system (X_1, X_2) which is translating at the final speed of the wedge with the origin coincident with the wedge vertex for $t \geq 0$.

The wedge velocity is assumed sufficiently large that a plane shock wave is attached at the wedge vertex. Since neither the wedge nor the gas have a characteristic length, one must suppose that for any time t a length defined, for example, by Vt will serve to define the scale of the flow. That is, the application of an impulsive acceleration to the wedge generates a wave pattern which grows uniformly with time. The character of this wave pattern becomes clear on further investigation. For a point on the body remote from the vertex, the impulsive acceleration generates a wave propagating in a direction normal to the wedge surface which induces a normal velocity $\Delta V \sin \alpha$ in the gas. For ΔV small, the propagation speed, relative to the gas, of this wave will be a_1, the acoustic velocity behind the initial shock wave. Simultaneously, a cylindrical wave will originate at the vertex at $t=0$. This wave, for small ΔV, will propagate radially from its center at speed a_1, and its center will convect along the wedge surface at the speed of the gas. A new shock wave will be formed at the vertex, corresponding to steady-state flow at the new wedge velocity. The wave geometry for a particular time is shown in Fig. 2. The dashed portion BC of the shock falls inside the circle defining the cylindrical wave; its shape is not known. If α is assumed

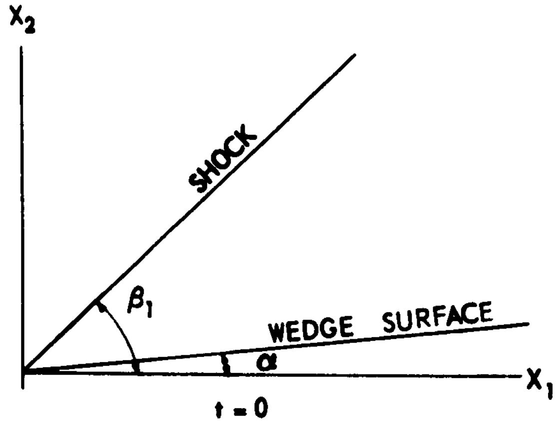

FIGURE 1

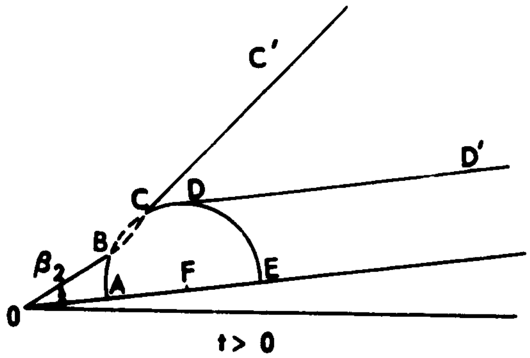

FIGURE 2

sufficiently small, then OB and CC′ will be tangent to the circle.* The geometry shown is for a relatively low Mach number so that DD′ is tangent to the circle. For sufficiently high Mach numbers, C and D coalesce and this point moves down the right hand side of the circle toward E as the Mach number increases.

3. TRANSFORMATION OF THE EQUATIONS

The momentum, continuity, and energy equations can be expressed for this problem as

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \qquad i, j = 1, 2 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0 \tag{2}$$

$$\frac{D}{Dt}\left(\frac{a^2}{\gamma - 1} + \frac{U_j U_j}{2}\right) = \frac{1}{\rho} \frac{\partial p}{\partial t} \quad , \tag{3}$$

where it is assumed that the fluid is a perfect gas.

The previous discussion suggests the transformation

$$\xi_i = \frac{x_i}{a_1 t} \tag{4}$$

If the flow variables are functions only of ξ_i, Eqs. (1), (2), and (3) become

*It was pointed out to the author by Professor Nicholas Rott that the angle BFC is proportional to $\alpha^{1/2}$ for small α, placing a more severe restriction on this angle than would at first seem necessary.

$$(U_j - a_1 \xi_j)\ \frac{\partial U_i}{\partial \xi_j} + \frac{1}{\rho}\ \frac{\partial p}{\partial \xi_i} = 0 \tag{5}$$

$$\frac{\partial\ (\rho\ U_i)}{\partial\ \xi_i} - a_1\ \xi_i\ \frac{\partial \rho}{\partial \xi_i} = 0 \tag{6}$$

$$(U_j - a_1\ \xi_j)\frac{\partial}{\partial \xi_j}\left(\frac{\gamma}{\gamma - 1}\ \frac{p}{\rho} + \frac{1}{2}\ U_i\ U_i\right) + \frac{a_1}{\rho}\ \xi_j\ \frac{\partial P}{\partial \xi_j} = 0 \tag{7}$$

We now define

$$\nu_i = \frac{U_i}{a_1} - W_i \tag{8}$$

$$\lambda = \frac{p - p_1}{\gamma\, p_1} \tag{9}$$

$$\sigma = \frac{\rho - \rho_1}{\rho_1} \tag{10}$$

$$\eta_i = \xi_i - W_i \quad , \tag{11}$$

where W_i is the (constant) vector Mach number behind the initial shock wave and p_1, ρ_1 are the pressure and density, respectively, behind the initial shock.

In the η_1, η_2 coordinate system the cylindrical wave ABCDE maps into half of the unit circle with F mapping into the origin. The waves and the wedge are stationary in this space (Fig. 3).

We formally assume that the restriction of small $\Delta V/V$ implies small values for ν_1, σ, and λ. Inserting Eqs. (8), (9), (10), and (11) into Eqs. (5), (6), and (7) and retaining only lowest order terms provides

$$\eta_j \frac{\partial \nu_1}{\partial \eta_j} - \frac{\partial \lambda}{\partial \eta_1} = 0 \tag{12}$$

$$\eta_j \frac{\partial \sigma}{\partial \eta_j} - \frac{\partial \nu_1}{\partial \eta_j} = 0 \tag{13}$$

$$\left[(\gamma-1) \; W_j - \eta_j\right] \frac{\partial \lambda}{\partial \eta_j} + \eta_j \; \frac{\partial \sigma}{\partial \eta_j} - \eta_j \; W_1 \, (\gamma-1) \; \frac{\partial \nu_1}{\partial \eta_j} = 0 \tag{14}$$

Eqs. (12) and (13) may be substituted into (14) to provide

$$\eta_1 \;\; \eta_j \; \frac{\partial \nu_j}{\partial \eta_1} - \frac{\partial \nu_1}{\partial \eta_j} = 0 \tag{15}$$

We consider Eqs. (12), (13), and (15) as our system, noting that Eqs. (12) and (15) may be considered independently of (13).

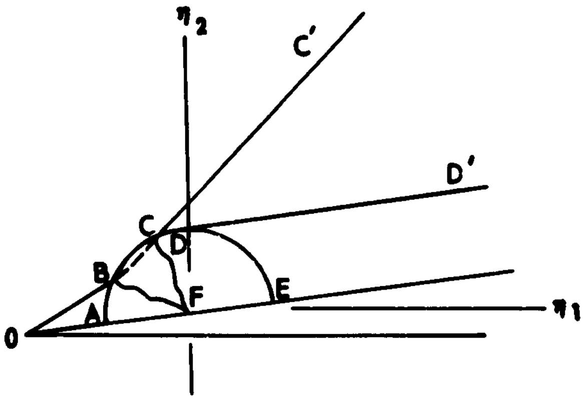

FIGURE 3

4. SOLUTION OF THE EQUATIONS

A number of manipulations can be performed on this system. It is shown in Ref. 4 that the characteristic directions for the systems are real outside the unit circle and imaginary inside. Since the solution outside the unit circle can be obtained from ordinary shock wave considerations, knowledge of the characteristic system does not seem to have much real significance except to define formally the unit circle as the boundary between the elliptic and hyperbolic regions of the plane. Examining Fig. 3, we see that the "streamlines" through points B and C must terminate at F, defining a triangular shaped region outside of which the fluid particles are acted upon by a plane shock only. The flow in regions ABF and in FCE will thus be irrotational. If the interaction BC is a "smooth" one, then the entropy gradient ds/dn and, hence, by Crocco's Theorem, the vorticity, will be of order ΔV in BCF.

If a velocity potential is postulated

$$\nu_1 = \frac{\partial \phi}{\partial \eta_1} \quad , \tag{16}$$

it can be shown that

$$(\eta_1^2 - 1)\frac{\partial^2 \phi}{\partial \eta_1^2} + 2\eta_1\,\eta_2 \frac{\partial^2 \phi}{\partial \eta_1 \partial \eta_2} + (\eta_2^2 - 1)\frac{\partial^2 \phi}{\partial {\eta_2}^2} = 0 \tag{17}$$

Solutions to this equation, using separation of variables, can be expressed as

$$\phi = \sum_0^\infty \left[A_n\, f(n, r) + B_n\, f(-n, r)\right]\left[C_n' \cos n\theta + D_n' \sin n\theta\right] \quad , \tag{18}$$

where $$r = (\eta_1^2 + \eta_2^2)^{1/2} \tag{19}$$

$$\theta = \tan^{-1}\left(\frac{\eta_2}{\eta_1}\right) \tag{20}$$

and $f(n, r)$ is expressed in terms of the hypergeometric function

$$f(n, r) = r^n \frac{F\left(\frac{n}{2}, \frac{n-1}{2}; \; n+1; r^2\right)}{F\left(\frac{n}{2}, \frac{n-1}{2}; \; n+1; 1\right)} \tag{21}$$

The presence of an infinity in $f(-n, r)$ at $r = 0$ suggests that

$$B_n = 0 \tag{22}$$

We may also choose, with no loss in generality,

$$A_n = 1 \quad , \tag{23}$$

giving

$$\phi = \sum_0^\infty f(n, r)\,(C_n' \cos n\theta + D_n' \sin n\theta) \tag{24}$$

These solutions apply in regions ABF and FCE.

5. THE PRESSURE EQUATION

With some manipulation, the velocities may be eliminated from Eqs. (12) and (15), providing

$$(\eta_1^2 + 1)\frac{\partial^2 \lambda}{\partial \eta_1^2} + 2\eta_1 \eta_2 \frac{\partial^2 \lambda}{\partial \eta_1 \partial \eta_2} + (\eta_2^2 + 1)\frac{\partial^2 \lambda}{\partial \eta_2^2} + 2\eta_1 \frac{\partial \lambda}{\partial \eta_1} + 2\eta_2 \frac{\partial \lambda}{\partial \eta_2} = 0 \quad (25)$$

or, in polar coordinates,

$$r^2 (r^2 + 1) \frac{\partial^2 \lambda}{\partial r^2} + r (2r^2 + 1) \frac{\partial \lambda}{\partial r} + \frac{\partial^2 \lambda}{\partial \theta^2} = 0 \quad (26)$$

The transformation

$$\tau = \frac{\sqrt{1 + r^2} - 1}{r} \quad (27)$$

transforms Eq. (26) into Laplace's equation

$$\tau^2 \frac{\partial^2 \lambda}{\partial \tau^2} + \tau \frac{\partial \lambda}{\partial \tau} + \frac{\partial^2 \lambda}{\partial \theta^2} = 0 \quad (28)$$

with general solution

$$\lambda = f(\tau e^{i\theta}) + g(\tau e^{-i\theta}) \quad (29)$$

6. BOUNDARY CONDITIONS

The flow properties in the region exterior to the unit circle are known, so that the values of pressure and velocity perturbations on segments AB, CD, DE, as well as the velocity and normal derivative of the pressure at the wedge surface are known.

On AB

$$\nu_1 = -\frac{\Delta V \sin\alpha \sin\beta_2}{\cos(\beta_2-\alpha)} = \nu_{21}$$

$$\nu_2 = \frac{\Delta V \sin\alpha \cos\beta_2}{\cos(\beta_2-\alpha)} = \nu_{22} \tag{30}$$

$$\lambda = \frac{p_2 - p_1}{\gamma p_1} = \lambda_2 \, ,$$

where β_2 and p_2 are, respectively, the wave angle for and the pressure behind the oblique shock OB.

On CD

$$\nu_1 = \lambda = 0 \tag{31}$$

On DE

$$\nu_1 = -\Delta V \sin^2\alpha$$

$$\nu_2 = \Delta V \sin\alpha \cos\alpha \tag{32}$$

$$\lambda = \frac{\Delta V}{a_1} \sin\alpha$$

On AE

$$\nu_1 \sin\alpha - \nu_2 \cos\alpha = 0$$

$$\frac{\partial\lambda}{\partial\eta} = 0 \tag{33}$$

On segment BC a difficulty arises in that the details of the shock interaction are not known. It is surmised that the shock segment BC is a smooth curve, matching slopes with OB at B and CC$'$ at C. A numerical scheme in which the shock shape is determined simultaneously with the solution of the flow equations, in the manner of current methods of solving the blunt body problem, would seem to be appropriate.

If the wedge angle α is assumed small, shocks OB and CC$'$ will, in the limit, be tangent to the circle, giving the geometry of Fig. 4. In this case the interaction occurs "outside" the unit circle and the conditions on BC are a simple superpostion of the conditions on AC and on BD. That is, $\lambda = \lambda_3 = \lambda_2 = 1/\gamma$ and $\nu_1 = \nu_{2i}$ on CB.

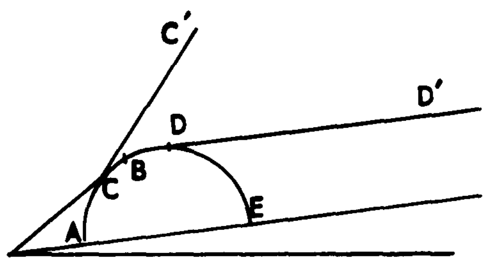

FIGURE 4

7. THE PRESSURE DISTRIBUTION

The solution for λ may now be written

$$\lambda = \lambda_2 - \frac{1}{\pi} \, \mathrm{Rl} \left\{ \frac{\Delta V \sin \alpha}{a_1} \left[i \log \left(\frac{z - ik}{z + ik} \right) + \frac{\pi}{2} \right] \right.$$

$$\left. - (1/\gamma + \lambda_2) \left[i \log \left(\frac{z - z_B}{z - \bar{z}_B} \right) + (\theta_B - \alpha) \right] + 1/\gamma \left[i \log \left(\frac{z - z_C}{z - \bar{z}_C} \right) + (\theta_C - \alpha) \right] \right\}, \tag{34}$$

where

$$
\begin{aligned}
k &= \sqrt{2}-1 & \bar{z}_B &= k e^{-i(\theta_B - \alpha)} \\
z &= r e^{i(\theta - \alpha)} & z_C &= k e^{i(\theta_C - \alpha)} \\
z_B &= k e^{i(\theta_B - \alpha)} & \bar{z}_C &= k e^{-i(\theta_C - \alpha)}
\end{aligned}
\qquad (35)
$$

This solution applies inside the circle ADE . Fig. 5 shows a plot of λ versus distance along wedge surface for different initial Mach numbers.

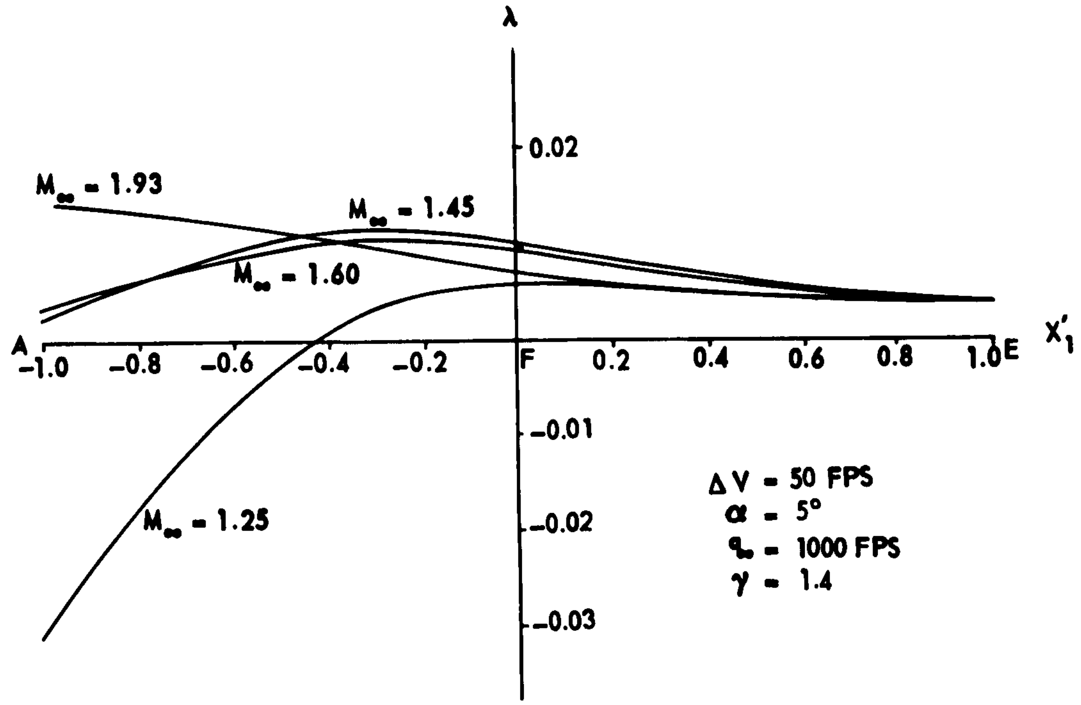

FIGURE 5

8. THE VELOCITY POTENTIAL

For the small angle case, the entropy gradient is of the order $\alpha^2 \Delta V$ and can be ignored. The coefficients of Eq. (24) can then be formally evaluated. We define

$$\psi = \theta - a$$

and write

$$\phi = \sum_{0}^{\infty} f(n,r) \left(C_n \cos n\psi + D_n \sin n\psi\right) \tag{36}$$

Since by Eq. (33) $\frac{\partial \phi}{\partial \psi} = 0$ at $\psi = 0$ and $\psi = \pi$, we require that ϕ be an even function of ψ, giving

$$D_n = 0$$

Eqs. (30), (31), and (32) can be expressed in terms of ψ as

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=1} = \begin{cases} \Delta V \sin a \cos \psi & 0 < \psi < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \psi < \psi_B \\ \nu_{21} \cos \psi + \nu_{22} \sin \psi & \psi_B < \psi < \pi \end{cases} \tag{38}$$

where only lowest order terms have been retained.

Formally

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=1} = \sum_{0}^{\infty} \frac{\partial f}{\partial r}(n,1)\, C_n \cos n\psi \quad ,$$

and since (Ref. 5)

$$F'\left(\frac{n}{2}, \frac{n-1}{2}; n+1; 1\right) = \frac{n(n-1)}{2} F\left(\frac{n}{2}, \frac{n-1}{2}; n+1; 1\right)$$

we have

$$\frac{\partial \phi}{\partial r}(1,\psi) = \sum_{1}^{\infty} n^2 C_n \cos n\psi \tag{39}$$

From Eq. (39), using standard methods,

$$C_n = -\frac{(-1)^{\frac{n}{2}}[1+(-1)^n]}{\pi n^2(n^2-1)} \Delta V \sin \alpha - \frac{\nu_{21}}{\pi n^2}\left[\frac{\sin (n+1)\psi_B}{n+1} + \frac{\sin (n-1)\psi_B}{n-1}\right]$$
$$+ \frac{\nu_{22}}{\pi n^2}\left[-\frac{2(-1)^n}{n^2-1} + \frac{\cos (n+1)\psi_B}{n+1} - \frac{\cos (n-1)\psi_B}{n-1}\right] \tag{40}$$

9. CONCLUSION

The pressure distribution and the potential function have been determined for the case of very small wedge angle. The solution for larger wedge angles could not be completely obtained owing to inability to specify the boundary values in the shock interaction region. However, the pressures to either side of the unit circle are constant and can be determined from ordinary shock considerations. It is felt that the nature of the pressure variation on the wedge surface inside of the unit circle is not markedly different for large angles from that given by the small angle solution.

The transient foredrag of a simple wedge wing can be estimated roughly by noting that for the extreme case of $M_\infty = 1.25$ the pressure coefficient to the right of E is approximately six percent greater than that to the left of A. The mean value of the transient foredrag coefficient would then be roughly half this value, that is, three percent of the final steady state foredrag coefficient. For the higher Mach numbers this value decreases, becoming slightly negative for $M_\infty = 1.93$.

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